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Error Rates for Square-Law Combining Receivers

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Lexington, Massachusetts

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ERROR RATES FOR SQUARE-LAW COMBINING RECEIVERS

S. L. BERNSTEIN

Group 66

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ABSTRACT

Error rates for square-law combining receivers operating with random phase, non-fading channels are calculated and discussed. Practical applications include the selection of code word parameters and specifying the required degree of diversity for receivers with multiple inputs. Combining losses and combining inefficiency are examined. Both binary and M-ary signalling are considered.

Accepted for the Air Force
Joseph R. Waterman, Lt. Col., USAF
Chief, Lincoln Laboratory Project Office

I. Introduction

In a number of cases of practical interest, communication receivers base their message decisions on the sums of the squares of matched filter envelopes. In particular, two cases are of immediate interest:

Case 1

One of M messages is to be transmitted. Each message waveform consists of a sequence of N_c channel symbols (or "chips"). Each channel symbol may be one of A orthogonal waveforms. The channel symbols are each received with a random (or unknown) phase shift. The signal structure is shown in Fig. 1.

Case 2

One of M messages is to be transmitted. Each waveform is constructed as in case 1, but is received by N_d separate receivers with random (or unknown) relative phase shifts.

II. Receiver Structure

The first case covers the waveform design problem for the random phase channel. The second case includes diversity combining. In either case the receiver forms the M sums:

$$R_m^2 = \sum_{n=1}^{N} e_{mn}^2 \qquad m = (0, M-1) \tag{1}$$

where e_{mn}^2, n = (1, N), are the squares of the matched filter envelopes corresponding to the m-th waveform. For case 1, $N = N_c$, the sum being taken over each codeword. For case 2, $N = N_d N_c$, the sum being over each code word and diversity channel. (A total of $A N_d N_c$ matched filters must be implemented.) Hence case 2 corresponds to post-detection combining.

The receiver then decides which message to announce by choosing the one with the largest R_m^2.

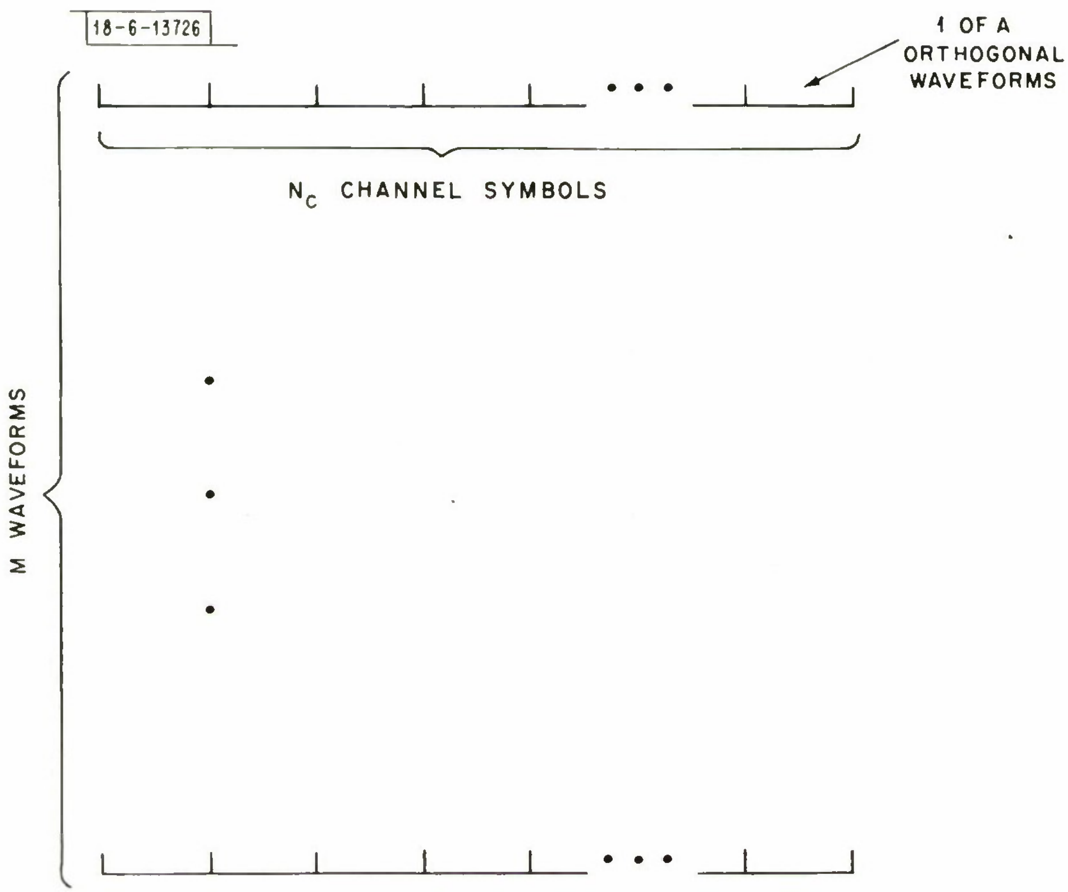

Fig. 1. Waveform structure.

In this report it will be assumed that the noise added to the signals before reception is white (over the band of interest) with double sided density $N_o/2$ and Gaussian. Also the noise is assumed independent and of equal density on each diversity channel.

The signals are assumed to be fixed in terms of received energy per channel, E_{mn}, but not necessarily equal. The total received signal energy per waveform, $\sum_{n=1}^{N} E_{mn}$, is assumed to be equal for all messages.

The receiver decision variables defined in (1) are not the optimum ones. The optimum decision variables would be:

$$R_m^{opt} = \sum_{n=1}^{N} \ln \left[I_o\left(\frac{2\sqrt{E_{mn}}}{N_o} e_{mn}\right)\right]. \tag{2}$$

However, computing error probabilities for this form of receiver must be done by numerical integration. The square-law combining (which R_m^{opt} reduces to for low E_{mn}/N_o) gives nearly equal error probabilities, and can be computed much more easily.

Examining R_m^{opt} shows that the channels with higher SNR's should be given greater weight, thus leading to maximal ratio combiners[1]. However, this refinement will not be studied in detail here, although the numerical results presented later can be used to show that this further departure from optimality will not usually cause very large degradations to performance.

III. Binary Signalling, M = 2

The case studied in greatest detail will be the one corresponding to one of only two equally likely messages being transmitted. This will include most of the important effects (e.g., combining loss) and also, through the use of the union bound, serves as the basis for understanding more complicated waveforms. Section IV discusses the case for $M > 2$.

A. Error Rate Evaluation

Lindsey [2] has derived the error probability for the case

considered here (and for Rician channels as well). His Eq. (51)* gives the probability of error as:

$$P_2(E;N) = [\tfrac{1}{2}]^N \exp[-\tfrac{1}{2}\frac{E_T}{N_o}] \sum_{n=0}^{N-1} \binom{N+n-1}{n} (\tfrac{1}{2})^n F(-n, N; -\tfrac{1}{2}\frac{E_T}{N_o}) \quad (3)$$

where $\frac{E_T}{N_o} = \sum_{n=1}^{N} \frac{E_{mn}}{N_o}$, the total energy to noise density ratio for the waveforms and $F(-n, N; x)$ is the hypergeometric function (which in this case reduced to an n-th degree polynomial) given as:

$$F(-n, N; x) = 1 + \frac{-n}{(N)(1!)} x + \frac{(-n)(-n+1)}{(N)(N+1)(2!)} x^2 + \ldots \quad (4)$$

Figure 2 plots $P_2(E;N)$ as a function of E_T/N_o for various values of N. This figure is most useful for discussing the waveform design problem of case 1 above. E_T/N_o is equal to E_B/N_o, the signal-to-noise ratio per bit taken over the entire code word and over all diversity channels.

Figure 3 plots $P_2(E;N)$ as a function of $(1/N)(E_T/N_o)$ the average signal-to-noise ratio per channel symbol or per diversity channel for various values of N. This figure is most useful for discussing the effects of adding additional degrees of diversity.

Both sets of curves have been accurately computed and plotted and are meant to be used for numerical purposes.

B. Discussion: Combining Loss

The curves of Fig. 2 clearly show what is commonly termed "combining loss". That is, for a fixed value of E_T/N_o, the error rate will increase as N increases. Thus for post-detection combining on a non-fading channel, once a certain amount of signal energy is split into a number of pieces, the full energy can never be recovered. (This is not necessarily true for pre-detection combining or for fading channels.)

Figure 4 plots the combining loss (in db) versus N for several error

*For internal consistency his N and M have been interchanged in meaning. Also there is a typographical error: his exponent should be divided by 2.

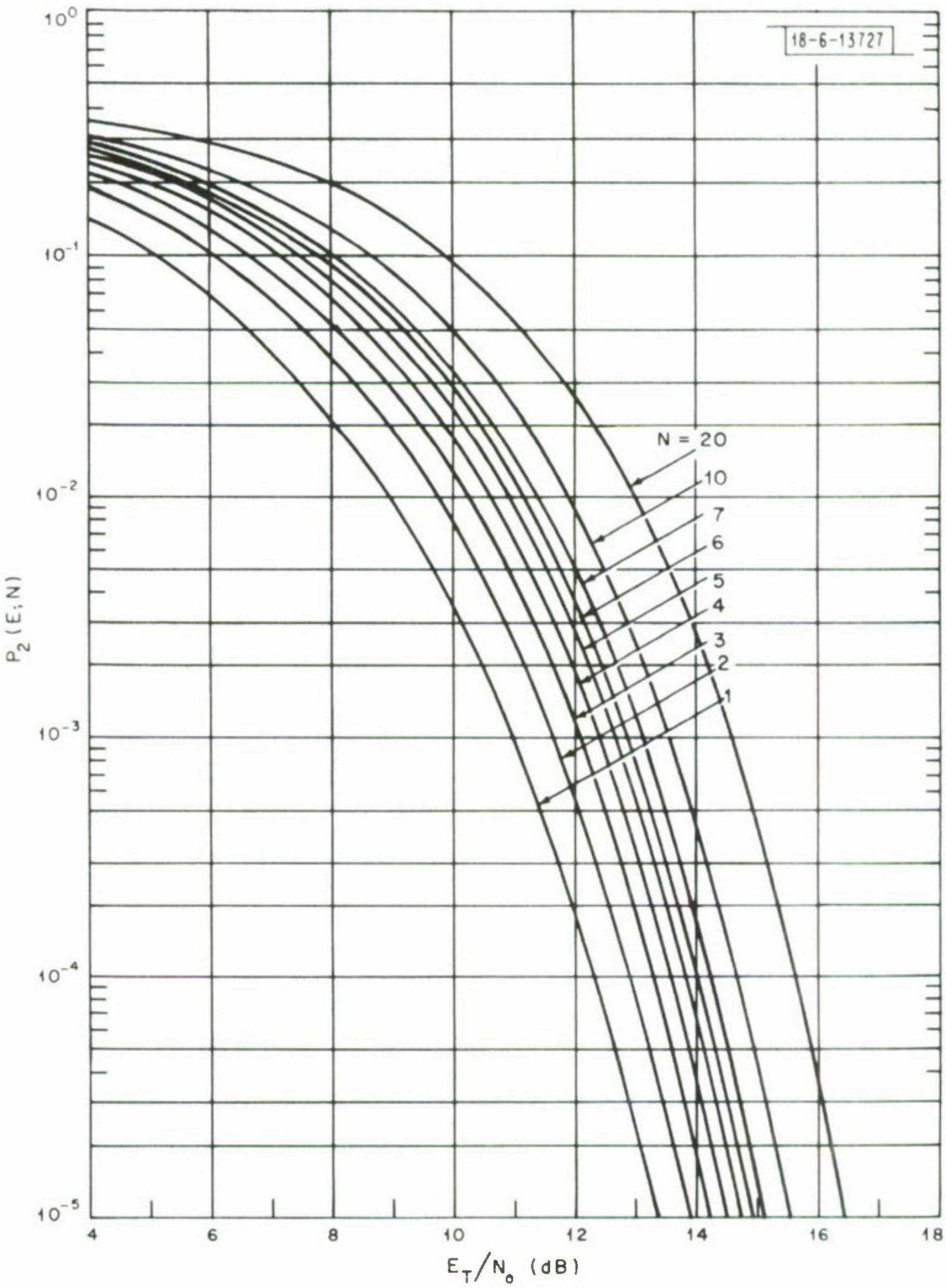

Fig. 2. Binary error probability vs total SNR per waveform.

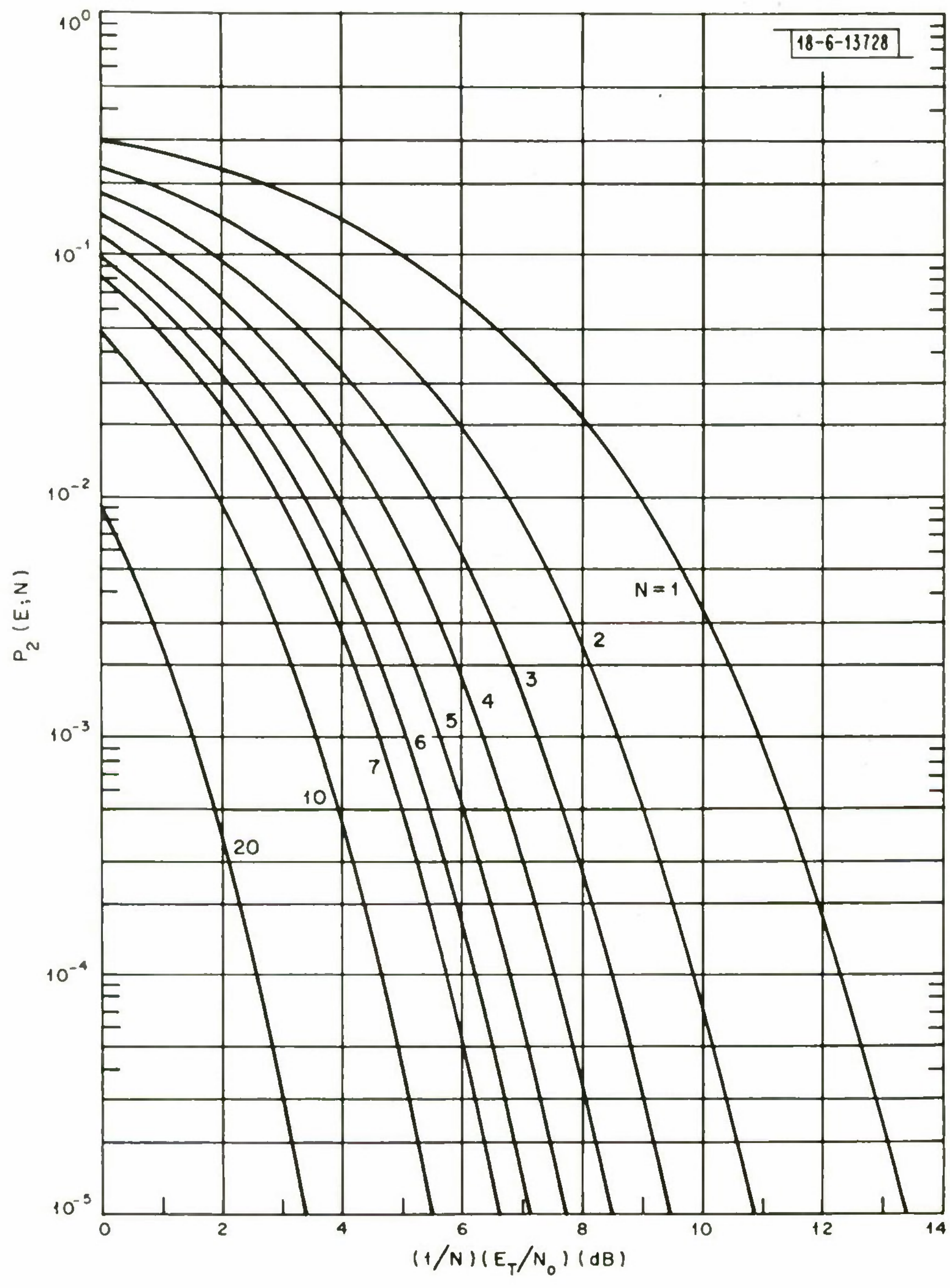

Fig. 3. Binary error probability vs average SNR per channel symbol.

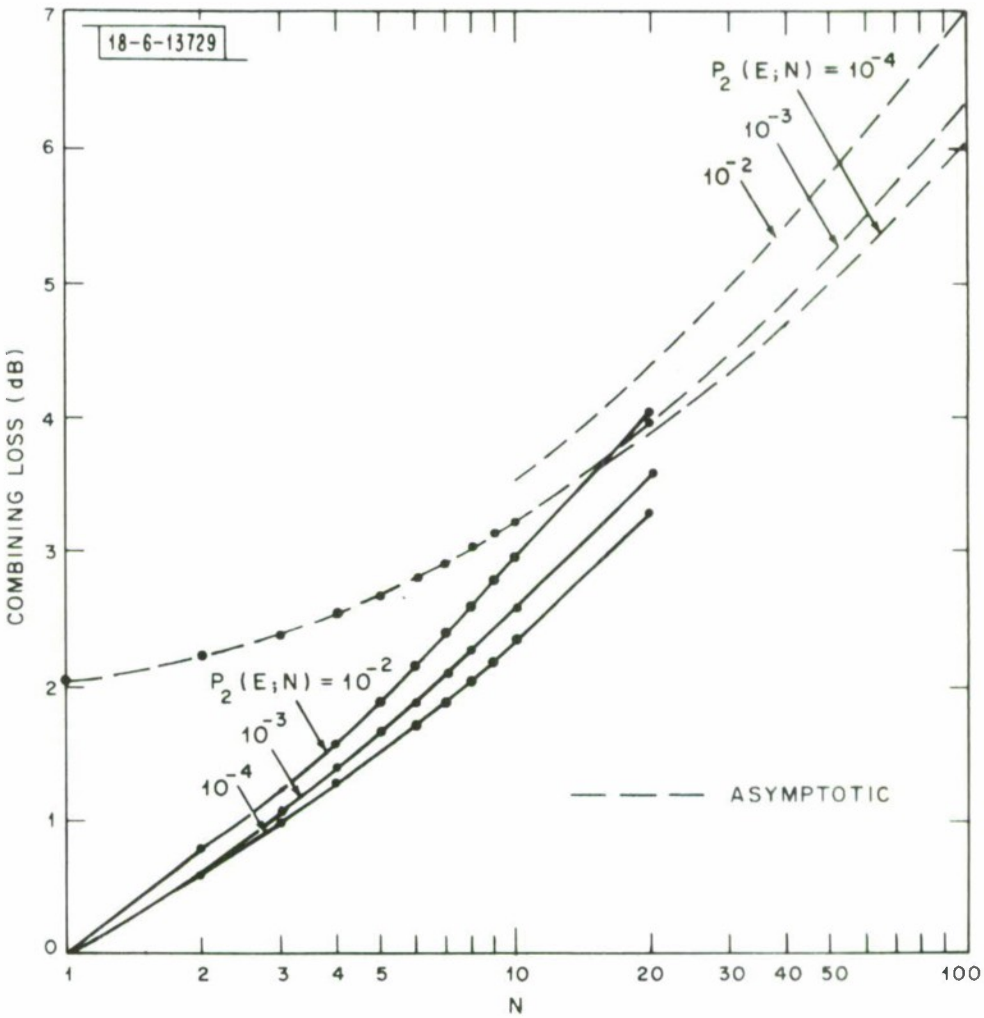

Fig. 4. Combining loss vs number of channel symbols.

rates. The combining loss is given by the difference between the (E_T/N_o)'s required to produce

$$P_2(E;N) = P_2(E;1) \tag{5}$$

It shows that the combining loss in general:

1) increases with N

and 2) decreases when E_T/N_o increases (or as $P_2(E;N)$ decreases)

C. Asymptotic Expressions: Combining Loss

A rather simple expression for the error rate, valid when N is large, can be derived from central limit theorem arguments. It is not difficult to show that:

$$\overline{e^2_{1n}} = E_{1n} + N_o \tag{6}$$

$$\mathrm{Var}\,[e^2_{1n}] = N_o^{\,2} + 2E_{1n}N_o \tag{7}$$

assuming message 1 was transmitted.

Thus when message 1 is transmitted the mean and variance of the decision variables are:

$$\overline{R_1^2} = E_T + NN_o \tag{8a}$$

$$\mathrm{Var}\,[R_1^2] = NN_o^2 + 2N_o\,E_T \tag{8b}$$

$$\overline{R_o^2} = NN_o \tag{8c}$$

$$\mathrm{Var}\,[R_o^2] = NN_o^2 \tag{8d}$$

The probability of error is given by:

$$P_2(E;N) = P_{rob}\,[\,R_1^2 - R_o^2 < 0]\,. \tag{9}$$

For large N both R_o^2 and R_1^2 will be approximately Gaussian and thus $(R_1^2 - R_o^2)$ will also be Gaussian. It's mean and variance will be:

$$\overline{(R_1^2 - R_o^2)} = E_T$$
$$\mathrm{Var}\,[R_1^2 - R_o^2] = 2(N N_o^2 + N_o E_T). \tag{10}$$

Hence for large N the error rate will be:

$$P_2(E;N) \approx Q\left[\sqrt{\frac{E_T^2}{2(N N_o^2 + N_o E_T)}}\right] \tag{11}$$

where

$$Q(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\frac{x^2}{2}}\, dx. \tag{12}$$

From Eqs. 11 and 12 an expression can be found for the E_T/N_o required to produce a given error rate (or α) as a function of N, namely:

$$\frac{E_T^2}{2(N N_o^2 + N_o E_T)} = \alpha^2 . \tag{13}$$

Solving,

$$\frac{E_T}{N_o} = \alpha^2 \left(1 + \sqrt{1 + \frac{2N}{\alpha^2}}\right) . \tag{14}$$

A short table for $Q(\alpha)$ is given below:

α	$Q(\alpha)$
1.28	10^{-1}
2.33	10^{-2}
3.10	10^{-3}
3.75	10^{-4}
4.30	10^{-5}
4.75	10^{-6}

Using Eq.(14) the combining loss (or the E_T/N_o) can be predicted for large N. The results are shown in the curves of Fig. 4 labeled 'asymptotic'. The agreement for N = 10 with actual combining loss is within about 1 dB, and at N = 20 the agreement is to within about 0.5 dB. For large N the results will be more accurate.

It is interesting to note that for large N:

$$N\left(\frac{1}{N}\frac{E_T}{N_o}\right)^2 = 2\alpha^2 . \tag{15}$$

Since $(1/N)(E_T/N_o)$ is the average SNR per channel symbol, Eq. (15) indicates that the ultimate combining loss is dominated by the quadratic small signal suppression effect associated with square-law detectors[3].

D. Discussion: Combining Inefficiency

The curves in Fig. 3 ($P_2(E;N)$ vs. SNR per channel) reveal a similar combining loss mechanism when combining diversity channels. For example, the curves for N = 1 and N = 2 are always less than 3 dB apart. The conclusion must be that combining two equal strength channels is less efficient than having a single channel with twice the energy. The same holds true for any other pair of curves. It can also be seen that the curves diverge as $(1/N)(E_T/N_o)$ increases. Ultimately the curves will be spaced at intervals of 10 log N db from the N = 1 curve since the probability of error will be dominated by the exponential in Eq. (3). Thus for large $(1/N)(E_T/N_o)$ the combining inefficiency approaches 0 db. This value is nearly reached for $(1/N)(E_T/N_o) \gtrsim 10$ dB.

E. Asymptotic Expression: Combining Inefficiency

Equation (13) may be solved a different way to give the required degree of diversity as a function of the SNR per channel as follows:

$$N = \frac{2\alpha^2[1 + (1/N)(E_T/N_o)]}{[(1/N)(E_T/N_o)]^2} \tag{16}$$

Hence for large N it can be predicted that the required number of channels varies nearly inversely with the square of the SNR per channel when the SNR per channel is small.

IV. M-ary Signalling

When M-ary signalling is considered, finding tractable and exact expressions for the probability of error is considerably more difficult than the binary case. Lindsey [2] has solved this problem (but does not give numerical results) for orthogonal waveform sets (A = M). When non-orthogonal waveforms are used, such as those derived from algebraic codes, bounding the error probability often gives satisfactory results.

The most familiar bound is the union bound [4], namely:

$$\begin{aligned} P(E|m_o) &= P_{rob}(R_1^2 > R_o^2 \text{ or } R_2^2 > R_o^2 \ldots R_{M-1}^2 > R_o^2 | m_o) \\ &\leq \sum_{m=1}^{M-1} P_{rob}(R_m^2 > R_o^2 | m_o) \end{aligned} \tag{17}$$

where it is assumed message 0 is sent. The union bound is always too conservative, predicting a required SNR greater than that actually needed.

A lower bound to the error rate can be found by considering errors occuring among triplets of signals (see [4], pg. 361):

$$\begin{aligned} P(E|m_o) \geq \sum_{i=1}^{M-1} P_{rob}(R_m^2 > R_o^2 | m_o) \\ - \sum_{i=2}^{M-1} \sum_{j=1}^{i-1} P_{rob}(R_i^2 > R_o^2,\ R_j^2 > R_o^2 | m_o) \ . \end{aligned} \tag{18}$$

The first term above is seen to be identical to the union bound, Eq. (17). The second term represents a first correction term to the union bound.

In the Appendix it is shown that when 3 message waveforms are mutually orthogonal (or for certain other special cases):

$$P_3(E;N) = P_{rob}(R_i^2 > R_o^2,\ R_j^2 > R_o^2 \mid m_o)$$

$$(i \neq j \neq 0)$$

$$\leq [\tfrac{1}{3}]^N \exp[-\frac{2E_T}{3N_o}] \sum_{n=0}^{2(N-1)} \binom{N+n-1}{n} (\tfrac{2}{3})^n F(-n;N,\frac{-E_T}{3N_o}). \qquad (19)$$

The above bound is shown plotted in Fig. 5 as a function of E_T/N_o for various values of N. (Note that a different scale from Fig. 2 has been used in order to accommodate the interesting regions.)

Together with the plots in Fig. 2 a reasonable estimate of the M-ary error probability can be found. If the M-signals are not mutually orthogonal, then the union bound (derived from Fig. 2) is all that can be generally easily applied. If the signals are mutually orthogonal and equally likely the following can be used to bound the error probability, P(E;N, M):

$$(M-1)P_2(E;N) - \frac{(M-1)(M-2)}{2} P_3^*(E;N) < P(E;N,M) < (M-1)P_2(E;N) \qquad (20)$$

where $P_3^*(E;N)$ is the bound in Eq. (19).

As an example, suppose N = 10 channel symbols and there are 16 orthogonal signals, M = 16. At a SNR per bit, E_B/N_o = 9 dB, the total waveform SNR will be E_T/N_o = 15 dB ($4E_B/N_o$). From Fig. 2 one finds $P_2(E;10) = 4.5 \times 10^{-5}$. From Fig. 5 one finds $P_3^*(E;10) = 3.5 \times 10^{-6}$. Hence from Eq. 20 the error rate is bounded by:

$$2.58 \times 10^{-4} < P(E;10,16) < 6.75 \times 10^{-4}$$

which is equivalent to an uncertainty of about ±.2 dB as seen from the slope of Fig. 2.

Generally speaking, the union bound increases in accuracy for lower error rates and/or smaller signal sets. When the estimated error rate is below about 10^{-3}, however, the accuracy appears to be quite good.

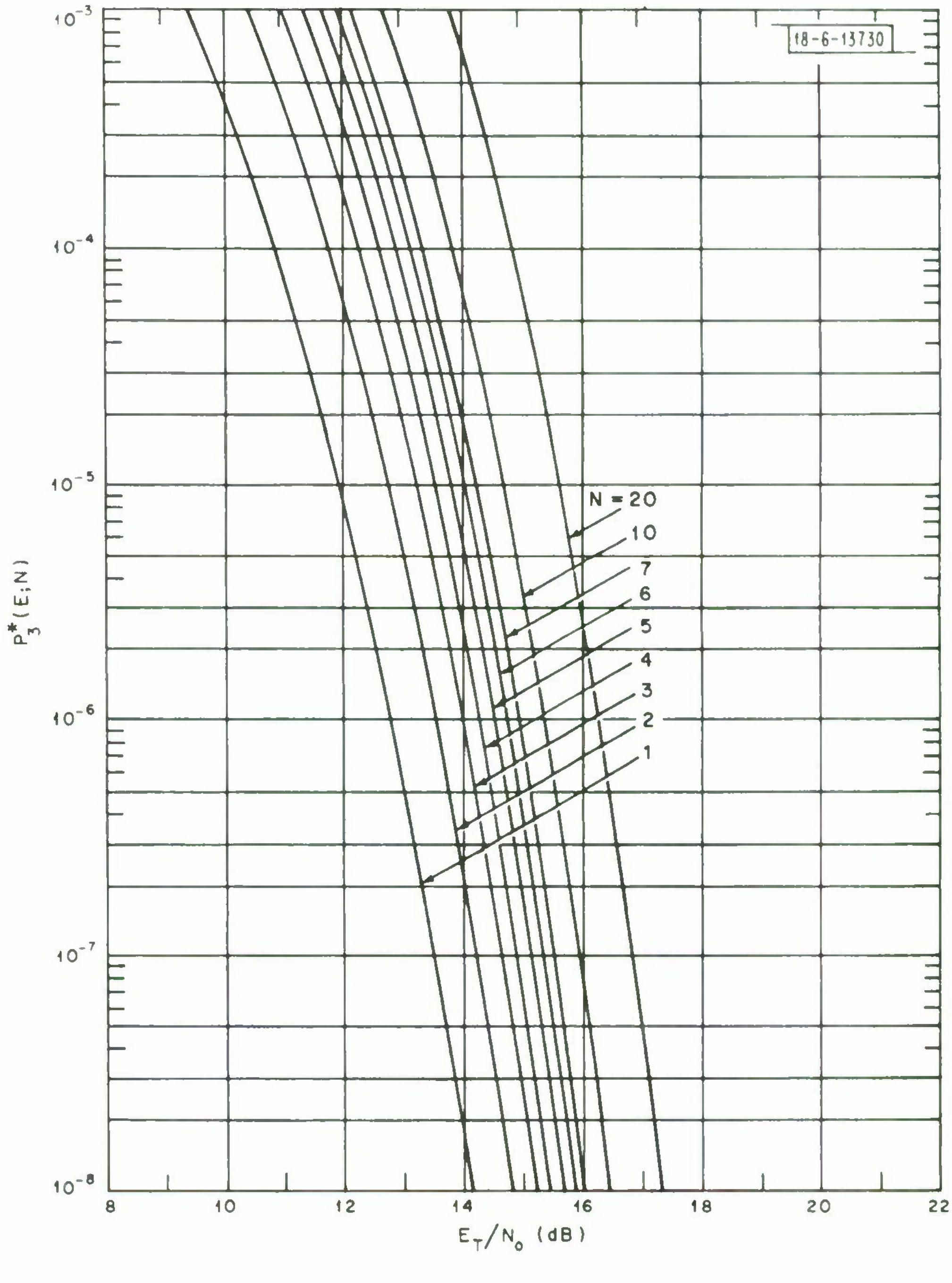

Fig. 5. Bound on triplet error probability vs total SNR per waveform.

The union bound approximation, depending solely on $P_2(E;N)$, indicates that the combining losses and inefficiencies in the M-ary case behave essentially the same as the binary case. Hence, the discussion and asymptotic approximation of the last section can be applied here also.

Finally, it should be noted that the curves of Figs. 2, 3 and 5 are dominated by the exponential term in their respective expressions at high SNR. Hence if one needs values for higher SNR's than those appearing on the curves, they may be extrapolated safely for a few more dB with the appropriate exponential factor.

APPENDIX

A Bound on $P_3(E;N)$

Lindsey's Eq. (41) can be expanded into the following two sums when the triplet of message waveforms are mutually orthogonal:

$$P_{rob}(R_1^2 > R_o^2 \text{ or } R_2^2 > R_o^2 | m_o)$$

$$= 2[\tfrac{1}{2}]^N \exp[-\tfrac{1}{2}\tfrac{E_T}{N_o}] \sum_{n=0}^{N-1} \binom{N+n-1}{n} (\tfrac{1}{2})^n F(-n;N,-\tfrac{E_T}{2N_o}) \qquad \text{(A-1)}$$

$$- [\tfrac{1}{3}]^N \exp[-\tfrac{2}{3}\tfrac{E_T}{N_o}] \sum_{n=0}^{2(N-1)} c_n \frac{(N+n-1)!}{(N-1)!} (\tfrac{1}{3})^n F(-n;N,\tfrac{-E_T}{3N_o})$$

where c_n is the coefficient of x^n in:

$$[\sum_{k=0}^{N-1} \frac{x^k}{k!}]^2 = \sum_{n=0}^{2(N-1)} c_n x^n. \qquad \text{(A-2)}$$

It is always true that:

$$P_{rob}(R_1^2 > R_o^2 \text{ or } R_2^2 > R_o^2 | m_o)$$

$$= P_{rob}(R_1^2 > R_o^2 | m_o) + P_{rob}(R_2^2 > R_o^2 | m_o) \qquad \text{(A-3)}$$

$$- P_{rob}(R_1^2 > R_o^2, R_2^2 > R_o^2 | m_o).$$

The first two terms in (A-3) form the sum of the binary error probabilities from Eq. (3) which in turn can be identified with the first summation in (A-1). Hence the second summation of (A-1) is equal to the last term of (A-3) which is $P_3(E;N)$. It now remains to find c_n.

It is not too difficult to show that:

$$c_n = \begin{cases} \sum\limits_{i=0}^{n} \frac{1}{i!} \cdot \frac{1}{(n-i)!} & n \le N-1 \\ \sum\limits_{i=n-(N-1)}^{N-1} \frac{1}{i!} \cdot \frac{1}{(n-i)!} & n > N-1 \,. \end{cases} \tag{A-4}$$

Looking back to the summation in (A-1), what is needed is:

$$c_n \frac{(N+n-1)!}{(N-1)!} = \begin{cases} \sum\limits_{i=0}^{n} \frac{1}{i!} \frac{1}{(n-i)!} \cdot \frac{(N+n-1)!}{(N-1)!} & n \le N-1 \\ \sum\limits_{i=n-(N-1)}^{N-1} \frac{1}{i!} \frac{1}{(n-i)!} \frac{(N+n-1)!}{(N-1)!} & n > N-1 \,. \end{cases} \tag{A-5}$$

If each term in the above sum is multiplied and divided by $(n!)$, each term is seen to be equal to $\binom{N+n-1}{n}\binom{n}{i}$. Hence the summations can be put into the form:

$$c_n \frac{(N+n-1)!}{(N-1)!} = \begin{cases} \binom{N+n-1}{n} 2^n & n \le N-1 \\ \binom{N+n-1}{n} \left[2^n - 2\sum\limits_{i=0}^{n-N} \binom{n}{i} \right] & n > N-1 \,. \end{cases} \tag{A-6}$$

Since an upper bound to $P_3(E;N)$ is sufficient to lower bound $P(E;N)$, the summation in (A-6) will be ignored. The leads to the upper bound given in Eq. (19) which is very similar in form to $P_2(E;N)$, thus simplifying the computations considerably.

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